FINAL EXAM: DIFFERENTIAL EQUATIONS

Duration: 3 hours **Total Marks: 50**

- Give necessary justification for all your arguments. If you are using any of the results proved in the class OR any named theorems state them clearly.

State true or false and explain. You may answer any 5, each question carries 2 marks. [2*5=10]

1. Let $\mathcal{S}$ denote the space of all C^1 solutions of a homogeneous first order linear partial differential equation. Then the set $\mathcal{S}$ is a vector space of one dimension.
2. The initial value problem $y'(t) = \sqrt{y}$ and $y(0) = \frac{1}{1000}$ admits a unique solution for $|t| < h$ for some $h > 0$.
3. There exist continuous functions p, q in $[-1, 1]$ such that the function $y(t) = t^3$ is a solution of the second order ode $y'' + p(t)y' + q(t)y = 0$ in $(-1, 1)$.
4. Let Ω be a connected open set in $\mathbb{R}^2$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\Delta u \geq 0$. Then $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.
5. The projected characteristics of a first order quasilinear PDE never intersect.
6. The Laplace equation on the upper half space, ie $\Delta u = 0$ on $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ and $u(x, 0) = g(x)$ for $x \in \mathbb{R}$ admits atmost one solution.

Answer any four from the given set of 5 questions. 10 marks each. [10*4=40]

1. (a) Let $v(x,t) = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}}\, dy$. Show that the function v is differentiable with respect to t for $t > 0$ and find the derivative. Deduce that $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}}\, dy$ is differentiable w.r.t t variable for all $t > 0$. [5]

   (b) Consider the heat equation $u_t = u_{xx}$ in $(x,t) \in (0,1) \times (0, \infty)$ with $u(0,t) = u(1,t) = 0$ and $u(x, 0) = 4x(1-x)$.

   (a) Show that $0 \leq u(x,t) \leq 1$ for all $t > 0$ and $0 \leq x \leq 1$.

   (b) Show that $u(x,t) = u(1-x, t)$ for all $t \geq 0$ and $0 \leq x \leq 1$. [2+3]

2. (a) Solve $xu_x + yu_y = u$ with the initial condition $u(x, 1) = h(x)$ by the method of characteristics.

   (b) Draw the projected characteristics of the above semilinear PDE passing through the initial curve $\Gamma = \{(x, 1) : x \in \mathbb{R}\}$. Do they ever intersect? [7+3]

3. Find the indicial equation and series solution about the regular singular point $x = 0$ of the following equation $xy'' + 4y' - xy = 0$. [10]

4. Let $\Omega := \{x \in \mathbb{R}^2 : |x| < R\}$ and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a non constant harmonic function in Ω. Define a function $\phi : (0, R) \to \mathbb{R}$ by

$$\phi(r) = \max_{|x|=r} u(x).$$

Show that ϕ is a strictly increasing function on the interval $(0, R)$. [10]

5. Using the separation of variables solve the following boundary value problem

$$\begin{aligned} u_t = u_{xx} - u \quad & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = 0, \quad & u(1,t) = 0, \quad \text{if } t \geq 0. \\ u(x,0) = \sin(4\pi x) \quad & \text{ if } x \in [0,1]. \end{aligned}$$

[10]